EUCLIDEAN MODELS FOR THE HYPERBOLIC DISK AND ITS GROUP OF MOTIONS

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Abstract. We show how to imbed isometrically the hyperbolic disk into a Hilbert space, the Bergman space of analytic functions in the disk square integrable with respect to Lebesgue measure. We construct a similar imbedding into the projective space of the Hardy space H^2, with the Fubini-Study metric. These imbeddings are generalized to the unit ball in $\mathbb{C}^n$ with the Bergman metric, Siegel's unit disk, and to hyperbolic three-space.

1. Introduction

We denote the unit disk in the complex plane by

$$\mathbb{D} = \{\tau \in \mathbb{C} \ : \ |\tau| < 1\}$$

with the hyperbolic metric

$$ds^2 = \frac{|d\tau|^2}{(1-|\tau|^2)^2} \, .$$

The group of isometries is the set of Möbius transformations with matrices

$$g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \quad , \quad |\alpha|^2 - |\beta|^2 = 1,$$

a group naturally isomorphic to $PSL(2,\mathbb{R})$. It is a classical problem to imbed this space isometrically in a Euclidean space of sufficiently high dimension. Hilbert proved in 1903 that in $\mathbb{R}^3$ there is no complete analytic surface of constant negative curvature, while positive answers were given by Blanusa in $\mathbb{R}^6$ in 1955, and by Rozendorn in $\mathbb{R}^5$. See [R] for a complete overview along these lines.

In an article that deserves to be better known [B], Ludwig Bieberbach presents a model of a non singular and complete surface in a separable Hilbert space, which is isometric to the hyperbolic unit disk $\mathbb{D}$. He does so by expanding the hyperbolic metric in a power series of $|\tau|^2$, as we now explain.

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Let $(e_n)_{n\geq 1}$ be an orthonormal basis in a complex Hilbert space H. For $\tau \in \mathbb{D}$ define

$$\varphi(\tau) = \sum_{n\geq 1} \frac{\tau^n}{\sqrt{n}}\, e_n \,.$$

If $\tau(t)$ is a differentiable curve in $\mathbb{D}$, its image in H is a curve $\gamma(t)$ with tangent vector

$$\gamma'(t) = \sum_{n\geq 1} \sqrt{n}\tau^{n-1}(t)\tau'(t)\, e_n \,.$$

The square of the norm of this vector in H is

$$\begin{aligned} \|\gamma'(t)\|^2 &= \sum_{n\geq 1} n|\tau(t)|^{2(n-1)}|\tau'(t)|^2 \\ &= \frac{|\tau'(t)|^2}{(1-|\tau(t)|^2)^2}\,. \end{aligned}$$

Bieberbach proves that this surface does not lie in any subspace of finite dimension, and essentially establishes the property that the whole group of Möbius isometries of the unit disk corresponds to restrictions on the surface of affine isometries of the ambient Hilbert space. We shall prove this in a concrete model for H in the first section.

Later, Calabi in his doctoral thesis generalizes this original idea of expanding a metric in a power series, to obtain isometric imbeddings of complex manifolds into infinite dimensional spaces [C1,2]. He does so by giving an algorithmic procedure to obtain a family of holomorphic functions $f_n(\tau)$ such that if the metric is

$$ds^2 = g(\tau,\bar\tau)|d\tau|^2$$

then

$$\sum_{n\geq 1} |f_n'(\tau)|^2 = g(\tau,\bar\tau)\,.$$

He also points out that analogous mappings exist to map $\mathbb{D}$ into a projective Hilbert space with the Fubini-Study metric, making the observation that affine isometries are the projection of unitary linear mappings.

The family of possible Hilbert spaces and its orthonormal basis is very large indeed, and one may wonder how does the imbedded surface looks in each of them. In this paper we claim that it is worthwhile to consider specific Hilbert spaces, namely

a) $A^2(\mathbb{D})$: the Bergman space of square integrable holomorphic functions in the unit disk, for Bieberbach's imbedding, and

b) $H^2(\mathbb{D})$: the Hardy space of holomorphic functions in the disk, square integrable in the boundary, for Calabi's imbedding in projective space.

The answers in Theorems 2.1 and 3.2 are surprisingly simple. It is the explicit formulas for these imbeddings that makes it possible to obtain similar results for

i) imbedding of the unit ball in $\mathbb{C}^n$ in Theorem 5.1,

ii) imbedding Siegel's generalized unit disk in Theorem 6.1, and

iii) imbedding of the hyperbolic three-space in Theorem 7.1.

It is clear to us that we are just scratching the surface of a very important and forgotten subject, with further connections to unitary representations of groups and to imbeddings of Riemann surfaces. These subjects lie outside the scope of this paper.

2. Imbedding the unit disk into Bergman space

We denote by $A^2(\mathbb{D})$ the Bergman space of analytic functions in the unit disk which are square integrable with respect to Lebesgue neasure. The scalar product we consider is

$$\langle f, h\rangle = \iint_{|z|<1} f(z)\overline{h(z)}\, dv(z)\,,$$

where $dv(z) = dz \wedge d\bar{z}/(-2\pi i)$. An orthonormal basis is the set of functions

$$e_n(z) = \sqrt{n}\; z^{n-1}\,,\ n \geq 1\,,$$

so that Bieberbach's imbedding

$$\varphi(\tau) = \sum_{n\geq 1} \frac{\tau^n}{\sqrt{n}}\, e_n$$

gives the following theorem.

Theorem 2.1. *The mapping $\varphi : \tau \to f_\tau$, where*

$$f_\tau(z) = \frac{\tau}{1-\tau z}$$

is an isometric imbedding of the unit disk into the Bergman space. The group of Möbius transformations of the disk into itself is induced on the imbedded surface by the affine isometries of Bergman space

$$T_g(f)(z) = f_{g(0)}(z) + f(\hat{g}(z))\hat{g}'(z)\,,$$

where $\hat{g}(z) = \bar{g}^{-1}(z)$.

Proof. We already explained in the introduction why is

$$\varphi(\tau) = \sum_{n\geq 1} \frac{\tau^n}{\sqrt{n}}\sqrt{n}\, z^{n-1} = \frac{\tau}{1-\tau z}$$

an isometric imbedding. We have to check the action of a Möbius transformation. We have that

$$\begin{aligned} f_{g(\tau)}(z) - f_{g(0)}(z) &= \frac{\alpha\tau+\beta}{(\bar\beta\tau+\bar\alpha)-(\alpha\tau+\beta)z} - \frac{\beta}{\bar\alpha-\beta z} \\ &= \frac{\tau}{((\bar\alpha-\beta z)-\tau(\alpha z-\bar\beta))(\bar\alpha-\beta z)} \\ &= \frac{\tau}{1-\tau\dfrac{\alpha z-\bar\beta}{\bar\alpha-\beta z}} \cdot \frac{1}{(\bar\alpha-\beta z)^2} \\ &= f_\tau(\hat g(z))\, \hat g'(z)\,, \end{aligned}$$

with

$$\hat g(z) = (\alpha z-\bar\beta)/(\bar\alpha-\beta z)\,.$$

Thus the affine transformation is a translation by $f_{g(0)}$ following a unitary operator in Bergman space.

□

Remark 1: In the image of the unit disk in Bergman space there are at least two different distance functions. One is the Riemannian distance between points, equal to the hyperbolic distance, but a second one is the euclidean distance $\|f_\tau - f_{\tau'}\|$. When pulled back to the hyperbolic disk, this gives a second distance invariant by Möbius transformations. Since $\|f_\tau\|^2 = \sum_{n\geq 1}\frac{|\tau|^{2n}}{n}$, it is not difficult to see that this new (Möbius invariant) distance is given by

$$\delta(\tau_1,\tau_2) = \sqrt{-\log(1-|\tau|^2)}\ ,\ \tau = \frac{\tau_1-\tau_2}{1-\tau_1\bar\tau_2}\,.$$

The functions f_τ allow us to extend the action not just of one-to-one, onto, functions on $\mathbb{D}$ but of analytic functions as well.

Theorem 2.2. *Let $u:\mathbb{D}\to\mathbb{D}$ be an n-to-1 analytic function, not necessarily onto, such that $u(0)=0$. Then the induced action of u on $\varphi(\mathbb{D})$ is the restriction of a bounded linear operator T on $A^2(\mathbb{D})$. Furthermore, for $f\in A^2(\mathbb{D})$*

$$(T^*f)(z) = f(\bar u(z))\bar u'(z)\,, \tag{2.1}$$

where $\bar u(z) = \overline{u(\bar z)}$ and T^* is the adjoint operator. Conversely, if $T: A^2(\mathbb{D})\to A^2(\mathbb{D})$ is a bounded linear operator such that $T(\varphi(\mathbb{D}))\subset\varphi(\mathbb{D})$, then T is induced as above by an analytic function $u:\mathbb{D}\to\mathbb{D}$.

Proof. We first prove the formula for $f \in A^2(\mathbb{D})$. We claim that if F satisfies $F' = f$ and $F(0) = 0$, then

$$\langle f, f_\tau \rangle = F(\bar\tau)\,.$$

Indeed, if

$$f(z) = \sum_{n\geq 0} a_n z^n = \sum_{n\geq 1} \frac{a_{n-1}}{\sqrt{n}}\, e_n\,,$$

then $\langle f, f_\tau \rangle = \sum_{n\geq 1} \frac{a_{n-1}}{n}\, \bar\tau^n$, which proves the claim.

We seek a linear operator T for which

$$T(f_\tau)(z) = \frac{u(\tau)}{1-u(\tau)z}\,.$$

If such an operator exists, then for any $f \in A^2(\mathbb{D})$

$$\langle\, T^* f, f_\tau \,\rangle \;=\; \langle f\,,\, \frac{u(\tau)}{1-u(\tau)z}\,\rangle\,,$$

which gives

$$\int_0^{\bar\tau} (T^*f)(x)dx = \int_0^{\overline{u(\tau)}} f(x)dx\,.$$

In this last expression we replace $\bar\tau = z$ and differentiate with respect to z to obtain (2.1). We use this equation to define T^*, and therefore, T itself.

Now

$$\begin{aligned} \|T^* f\|^2 \;&=\; \iint_{\mathbb{D}} |f|^2(\bar u(z))|\bar u'(z)|^2\, dv(z) \\ &\leq\; n \iint_{u(\mathbb{D})} |f|^2(u) dv(u) = n\|f\|^2\,, \end{aligned}$$

proving that $\|T^*\| \leq \sqrt{n}$, and therefore that T is bounded as well.

For the converse, let T be a linear bounded operator such taht $T(\varphi(\mathbb{D})) \subset \varphi(\mathbb{D})$. Then we can define a function u via

$$T(f_\tau)(z) = \frac{u(\tau)}{1-u(\tau)z}\,,$$

and, evaluating at $z = 0$,

$$u(\tau) = T(f_\tau)(0)\,.$$

Now

$$f_\tau = \sum_{m\geq 1} \frac{\tau^n}{\sqrt{n}}\, e_n\,,$$

with convergence in L^2 norm, so that

$$u(\tau) = \sum_{n\geq 1} \frac{\tau^n}{\sqrt{n}}\, T(e_n)(0)\,.$$

Since $|T(e_n)(0)| \leq \|T(e_n)\|_2 \leq \|T\|$, this series converges for $|\tau| < 1$, so that u is an analytic function with $u(0) = 0$.

□

We can also give a converse to Theorem 2.1.

Theorem 2.3. *Let $T : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ be a continuous, one to one affine transformation such that $T(\varphi(\mathbb{D})) = \varphi(\mathbb{D})$. Then $T = T_g$ for some $g \in PSL(2, \mathbb{R})$.*

Proof. Let $T(f) = t + A(f)$, where A is linear and continuous. Here $T(0) = t \in \varphi(\mathbb{D})$, so that $t = \tau_0(1 - \tau_0 z)^{-1}$ for some $\tau_0 \in \mathbb{D}$. Now,

$$T(f_\tau)(z) = \frac{g(\tau)}{1 - g(\tau)z}$$

for some function $g : \mathbb{D} \to \mathbb{D}$. By evaluating at $z = 0$ we have

$$g(\tau) = T(f_\tau)(0)\,.$$

Since

$$f_\tau = \sum_{n \geq 1} \frac{\tau^n}{\sqrt{n}}\, e_n\,,$$

with convergence in L^2 norm,

$$T(f_\tau) = t + \sum_{n \geq 1} \frac{\tau^n}{\sqrt{n}}\, A(e_n)\,,$$

so that

$$\frac{g(\tau + h) - g(\tau)}{h} = \frac{1}{h} \sum_{n \geq 1} \frac{(\tau + h)^n - \tau^n}{\sqrt{n}}\, A(e_n)(0)\,.$$

Thus

$$\left| \frac{g(\tau + h) - g(\tau)}{h} - \sum_{n \geq 1} \sqrt{n}\tau^{n-1}\, A(e_n)(0) \right| \leq |h| \|A\| \sum_{n \geq 1} \frac{(n+1)n}{2} (|\tau| + |h|)^{n-1}\,,$$

proving that g is holomorphic. Since $T(\varphi(\mathbb{D})) = \varphi(\mathbb{D})$, g is onto, and since T is one-to-one, g is one-to-one. Therefore $T = T_g$ on $\varphi(\mathbb{D})$. To prove that they coincide everywhere we show the following:

Let $\{\tau_n\}$ be a sequence in $\mathbb{D}$ converging to $\sigma \in \mathbb{D}$. If $f \in A^2(\mathbb{D})$ satisfies $\langle f, f_{\tau_n} \rangle = 0$ for all n then $f \equiv 0$.

Indeed, as in the beginning of the proof of Theorem 2.2,

$$\langle f, f_{\tau_n} \rangle = F(\bar{\tau}_n)\ , \quad F' = f\ \ , \ F(0) = 0\,.$$

Since an analytic function vanishing on a non discrete set vanishes everywhere, this shows that $F \equiv 0$, $f \equiv 0$.

□

3. Imbedding the unit disk in projective space

As stated by Calabi in his thesis, [C1], and later in his paper on Riemann surfaces, [C2], the problems of isometric complex analytic imbeddings of metric surfaces can be solved by mapping the surfaces into projective spaces with the Fubini-Study metric. We will now see how this approach leads to an imbedding of the unit disk into the proyective space of H^2, the space of analytic functions L^2 on the boundary, instead of into Bergman space.

The Fubini-Study metric. Let M be a complex manifold of dimension n. A hermitian metric on M is given by a positive definite inner product

$$(\ ,\ )_z \,:\, T'_z(M) \otimes \overline{T'_z(M)} \to \mathbb{C}$$

of the holomorphic tangent space at $z \in M$ such that for local coordinates (z_i) the functions

$$h_{ij}(z) = (\partial/\partial z_i, \partial/\partial z_j)_z$$

are C^∞. The real part of this hermitian metric defines a Riemannian metric on M whose norm is given by

$$\mathrm{Re}\left\{\sum h_{ij}(z) v_i \bar{v}_j\right\}$$

at T_z on the tangent vector (v_i). The Fubini-Study metric is defined on complex projective space $\mathbb{P}^n$ as follows. Let $Z = (z_0, \dots z_n)$ be coordinates in $\mathbb{C}^{n+1}$ and $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ the standard proyection map. The metric is defined by the form

$$w = i/2\partial\bar{\partial} \log \|Z\|^2$$

which, in the neighborhood with $z_0 \neq 0$ and coordinates $(w_1, \dots, w_n)$ where $w_i = z_i/z_0$, is given explicitely by

$$w = (i/2\pi)\left[\frac{\sum dw_i \wedge d\bar{w}_i}{1 + \sum w_i \bar{w}_i} - \frac{(\sum \bar{w}_i dw_i) \wedge (\sum w_i \bar{w}_i)}{(1 + \sum w_i \bar{w}_i)^2}\right],$$

that is

$$h_{ij} = \frac{(1 + \sum_{k=1}^{n} |w_k|^2)\delta_{ij} - w_i \bar{w}_j}{(1 + \sum_{k=1}^{n} |w_k|^2)^2}$$

(see, *e.g.* [G], Chapter 0). Following Calabi, we shall use this last expression for the metric with $n = +\infty$ in a separable Hilbert Space.

Consider then a complex Hilbert space H with an orthonormal basis $(e_n)_{n\geq 0}$. We define on the unit disk $\mathbb{D}$ in $\mathbb{C}$ the mapping $\varphi : \mathbb{D} \to H$ via

$$\varphi(\tau) = \sum_{n\geq 0} \tau^n\, e_n\,, \tag{3.1}$$

that is, we do not divide by $\sqrt{n}$ as in Bieberbach's imbedding.

The image is fully contained in the neighborhood where $z_0 \neq 0$ and the corresponding map $\bar{\varphi} : \mathbb{D} \to \mathbb{P}(H)$ is given by composition with the standard projection map $\pi : H - \{0\} \to \mathbb{P}(H)$.

Theorem 3.1. *The pull-back of the Fubini-Study metric on $\bar{\varphi}(\mathbb{D})$ is the hyperbolic metric on $\mathbb{D}$*

Proof. For a curve $\tau = \gamma(t) = x(t) + iy(t)$ in $\mathbb{D}$ the square of the length of $\gamma'(t)$ is

$$(x')^2 + (y')^2/(1-|\tau|^2)^2$$

in the hyperbolic metric. We compute the square of the norm in $\mathbb{P}(H)$ of the tangent vector to π o $\varphi(\tau)$. Here then $w_n = \tau^n,\ w_n' = n\tau^{n-1}(x' + iy'),\ n \geq 1$ and $1 + \sum_{n\geq 1} |w_n|^2 = (1-|\tau|^2)^{-1}$. Therefore

$$\sum_{n,m\geq 0} h_{n,m} w_n' \overline{w'}_m =$$
$$\sum \left[\delta_{nm} nm\tau^{n-1}\bar{\tau}^{m-1}(1-|\tau|^2) - nm\tau^{2n-1}\bar{\tau}^{2m-1}(1-|\tau|^2)^2\right]\left((x')^2 + (y')^2\right)\ .$$

The first sum is

$$\begin{aligned}(1-|\tau|^2)\textstyle\sum_{n\geq 1} n^2|\tau|^{2(n-1)} &= (1-|\tau|^2)(1+|\tau|^2)(1-|\tau|^2)^{-3} \\ &= (1+|\tau|^2)(1-|\tau|^2)^{-2}\,,\end{aligned}$$

while the second sum gives

$$|\tau|^2(1-|\tau|^2)^2 \sum_{n,m\geq 1} nm|\tau|^{2(n-1)}|\tau|^{2(m-1)} = |\tau|^2(1-|\tau|^2)^{-2}\,.$$

The difference is therefore the required hyperbolic length.

□

The imbedding into H^2. The Hardy space H^2 is the class of analytic functions f in the unit disk for which the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in L^2 norm as $r \to 1$. This space is isomorphic to the space of functions L^2 on the unit circle such that

$$\int_{-\pi}^{+\pi} f(\theta)e^{in\theta}d\theta = 0\,,\quad n = 1, 2, 3, \dots\ ,$$

the scalar product is

$$\langle f, g\rangle = \int_{|z|=1} f(z)\overline{g(z)}|dz|\,,$$

and in this space the functions $(z^n)_{n\geq 0}$ form an orthonormal basis. The mapping φ in (3.1) gives

$$\varphi(\tau) = \sum_{n\geq 0} \tau^n z^n = \frac{1}{(1-\tau z)}\,. \tag{3.2}$$

As usual, we denote the class of a function by $[f]$ in projective space.

Theorem 3.2. *The mapping $\tau \to [\varphi_\tau]$, where $\varphi_\tau(z) = (1-\tau z)^{-1}$, is an isometric imbedding of the hyperbolic unit disk into $\mathbb{P}(H^2)$. The group of isometries of the disk is given by the restriction to the imbedded surface of the (projection of) the unitary operators*

$$(Uf)(z) = f((\alpha z+\beta)(\bar\beta z+\bar\alpha)^{-1})(\bar\beta z+\bar\alpha)^{-1}\,.$$

Proof. The first part was already established. For the second part, it is clear that U is a unitary operator in H^2 and we have just to compute:

$$\begin{aligned}
\varphi_{(\alpha\tau+\beta)(\bar\beta\tau+\bar\alpha)^{-1}}(z) &= \frac{1}{1-\dfrac{\alpha\tau+\beta}{\bar\beta\tau+\bar\alpha}z} \\
&= \frac{\bar\beta\tau+\bar\alpha}{(\bar\beta-\alpha z)\tau+(\bar\alpha-\beta z)} \\
&= \frac{\bar\beta\tau+\bar\alpha}{1-\dfrac{\alpha z-\bar\beta}{-\beta z+\bar\alpha}\tau}\cdot\frac{1}{-\beta z+\bar\alpha}\,.
\end{aligned}$$

Therefore

$$[\varphi_{g(\tau)}](z) = [\varphi_\tau](\widehat{g}(z))\widehat{g}'^{1/2}(z)\,,$$

where $\widehat{g}(z) = \bar{g}^{-1}(z)$.

∎

Remark 2: It is interesting to consider the spherical distance from $[\varphi_\tau]$ to $[\varphi_{\tau'}]$ as it gives a different metric in $\mathbb{D}$ invariant by Möbius transformations. A calculation gives that

$$\delta(\tau_1,\tau_2) = \arctan\frac{|\tau|}{\sqrt{1-|\tau|^2}} \quad , \ \tau = \frac{\tau_1-\tau_2}{1-\tau_1\bar\tau_2}\,.$$

This imbedding leads to a natural map not just of the disk but of all $SL(2,\mathbb{R})$ as we explain in the next section.

4. The unit tangent bundle, the Sasaki metric and $SL(2,\mathbb{R})$

There exist several correspondences between spaces, metrics and groups which we now make precise.

1. The bijection $(z-i)/(z+i)=\tau$ gives a conformal mapping from the upper-half-plane $\mathbb{H}$ to the unit disk $\mathbb{D}$. Under this correspondence the group

$$SL(2,\mathbb{R})=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \ : \ ad-bc=1,\ a,b,c,d\in\mathbb{R}\right\}$$

is isomorphic to the group of matrices

$$\left\{\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \ : \ |\alpha|^2-|\beta|^2=1\right\}$$

with $\alpha=(a+d)/2+i(b-c)/2$ and $\beta=(a-d)/2-i(b+c)/2$. We shall refer to this second group also as $SL(2,\mathbb{R})$; the group of isometries of either space is $PSL(2,\mathbb{R})=SL(2,\mathbb{R})/(\pm I)$.

2. The unit tangent bundle of the hyperbolic disk is given as

$$S\mathbb{D}=\{(\tau,u)\ :\ |\tau|<1\ ,\ u=e^{i\theta}(1-|\tau|^2)\ ,\ 0\le\theta<2\pi\}\,.$$

The identification of $PSL(2,\mathbb{R})$ with $S\mathbb{D}$ is given by

$$g\rightarrow(g(0),g'(0))$$

or, more explicitly, by the set of equations

$$\begin{aligned} \tau &= \beta/\bar\alpha \quad , \quad e^{i\theta} = \alpha/\bar\alpha\,, \\ \alpha &= e^{i\theta/2}(1-|\tau|^2)^{-1/2} \quad , \quad \beta = e^{-i\theta/2}\tau(1-|\tau|^2)^{-1/2}\,. \end{aligned}$$

3. In any Riemannian metric space M there exists a natural metric on the tangent bundle TM, the so called Sasaki metric (see [P], Chapter 1, for full details). In the unit disk $\mathbb{D}$ this metric on $T\mathbb{D}$ is given by the quadratic form at $(\tau,u)=p$,

$$\frac{1}{(1-|\tau|^2)^2}\left(|v|^2+|w+\frac{2\bar\tau}{(1-|\tau|^2)}uv|^2\right)\,,$$

where (v,w) is the tangent vector in $T_pT\mathbb{D}$. When restricted to $S\mathbb{D}$ this gives the metric

$$\frac{1}{(1-|\tau|^2)^2}(|v|^2+|w+2\bar\tau e^{i\theta}v|^2)\,,$$

and it is important to us that if $g(\tau)=(\alpha\tau+\beta)/(\bar\beta\tau+\bar\alpha)$ is an isometry so is

$$Tg:T\mathbb{D}\rightarrow T\mathbb{D},\ (Tg)(\tau,u)=(g(\tau),g'(\tau)u)\,.$$

Indeed, the quadratic form is preserved by

$$d_p(Tg)(v,w)=(g'(\tau)v,g''(\tau)uv+g'(\tau)w)\,.$$

Furthermore $SL(2,\mathbb{R})$ as an étalé covering of $PSL(2,\mathbb{R}) \simeq S\mathbb{D}$ inherits the same metric. We summarize our remarks in the next theorem.

Theorem 4.1. *The following diagrams commute, where the horizontal arrows are isometric imbeddings and the vertical arrows are the natural projections*

$$\begin{array}{ccc} SL(2,\mathbb{R}) & \xrightarrow{\psi} & H^2 \\ \downarrow & & \downarrow \\ PSL(2,\mathbb{R}) \simeq S\mathbb{D} & \longrightarrow & H^2/(\pm 1) \\ \downarrow & & \downarrow \\ \mathbb{D} & \xrightarrow{\varphi} & \mathbb{P}(H^2) \end{array}$$

Here $\psi\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \dfrac{1}{-\beta z + \bar{\alpha}}$, *as a function of* z *in* $\mathbb{D}$ *and* $SL(2,\mathbb{R})$ *carries the Sasaki metric.*

Proof. We already have defined $\varphi(\tau) = [(1-\tau z)^{-1}]$ and commutativity follows from

$$\left[\frac{1}{-\beta z + \bar{\alpha}}\right] = \left[\frac{1}{1 - \beta/\bar{\alpha} z}\right]$$

in projective space, as functions of z, and from $\tau = \beta/\bar{\alpha}$.

In $SL(2,\mathbb{R})$ left multiplication by $g = \begin{pmatrix} \gamma & \delta \\ \bar{\delta} & \bar{\gamma} \end{pmatrix}$ gives under ψ the unitary operator in H^2

$$\begin{aligned} Uf(z) &= f(\widehat{g}(z))\widehat{g}'^{1/2}(z) \\ &= f\left(\tfrac{\gamma z - \bar{\delta}}{-\delta z + \bar{\gamma}}\right) \frac{1}{-\delta z + \bar{\gamma}}\,. \end{aligned}$$

Therefore, to establish the theorem it is enough to check the isosmetry at one point in $S\mathbb{D}$. The Sasaki metric at $(0,1)$ on tangent vector (v,w) is $|v|^2 + |w|^2$.

On the other hand, at $t = 0$, the tangent vector to

$$\frac{1}{-\beta(t)z + \bar{\alpha}(t)}$$

is

$$\beta'(0)z - \overline{\alpha'}(0)\,,$$

when $\beta(0) = 0$, $\alpha(0) = 1$, and the H^2 norm is $|\beta'(0)|^2 + |\alpha'(0)|^2$. Since the curve is adapted to the tangent vector, $\beta'(0) = w$, $\alpha'(0) = v$, and the result follows.

□

We end this section with the observation that, in terms of the upper-half-plane model, the imbedding is

$$\mathbb{H} \to \mathbb{P}H^2(\mathbb{H}), \;\; t \to f_t(z) = \left[\frac{1}{z+t}\right] .$$

5. Imbedding the unit ball in $\mathbb{C}^n$

For all necessary results in this section, the reader is referred to Chapters 1,2 in [Kr]. Let $\Omega \subset \mathbb{C}^n$ be a domain. The Bergman space $A^2(\Omega)$ is defined as the space of all holomorphic functions in Ω which are square integrable (with respect to ordinary Lebesgue measure). It is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_\Omega f(z) g(\bar{z}) dV(z) .$$

For each $\zeta \in \Omega$ fixed, the evaluation functional

$$f \to f(\zeta)$$

is linear and continuous. The Bergman kernel K is defined by

$$f(\zeta) = \int_\Omega K(\zeta, z) f(z) dV(z) ,$$

it satisfies $K(\zeta, z) = \overline{K(z, \zeta)}$. If g is an automorphism of Ω then

$$K(g(\zeta), g(z)) Jg(\zeta) \overline{Jg(z)} = K(\zeta, z) .$$

The Bergman metric is the Hermitian metric on Ω defined by

$$g_{ij}(z) \;=\; \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, \bar{z}) ,$$

a metric that is invariant under the automorphisms of Ω. For the unit ball $\mathbb{B}^n$, the Bergman kernel and metric are given explicitly by the formulas

$$k(z, \zeta) \;=\; \frac{n!}{\pi^n} \frac{1}{(1 - z^t \bar{\zeta})^{n+1}} ,$$

and

$$g_{ij}(z) \;=\; \frac{n+1}{(1-|z|^2)^2} \left[(1-|z|^2)\delta_{ij} + \bar{z}_i z_j\right] .$$

It makes no harm to add a convenient constant and redefine

$$g_{ij}(z) \;=\; \frac{n!}{\pi^n} \frac{n+1}{(1-|z|^2)^2} \left[(1-|z|^2)\delta_{ij} + \bar{z}_i z_j\right] .$$

The automorphisms of $\mathbb{B}^n$ are given by the transformations of the form

$$g(z) \;=\; \frac{Az+B}{Cz+D} , \tag{5.1}$$

where A, B, C, D are matrices of order is $n\times n$, $n\times 1$, $1\times n$, and 1×1, respectively, which satisfy

$$A^*A - C^*C = I \ \ , \ \ |D|^2 - B^*B = 1 \ \ , \ \ A^*B - C^*D = 0\,.$$

Here * denotes conjugate transpose, as usual. The matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

preserves the quadratic form

$$F(w,w) = -\sum_{j=1}^{n} |w_j|^2 + |w_{n+1}|^2\,.$$

However, since in the action of g we may multiply all coefficients in the matrices by a constant, we will normalize and assume that $det(M) = 1$ (the above identities for A, B, C, D changing accordingly). With this normalization we have that

$$Jg(z) = \frac{1}{(Cz+D)^{n+1}}\,.$$

As in Section 3, for a separable Hilbert space H we consider the Fubini-Study metric in $\mathbb{P}(H)$ given in terms of a basis $(e_n)_{n\geq 0}$ and coordinates $(1, w_1, w_2, \ldots)$ by

$$h_{ij} = \frac{(1+\sum |w_i|^2)\delta_{ij} + w_i\bar{w}_j}{(1+\sum |w_i|^2)^2}\,.$$

Theorem 5.1. *The function $\tau \to [\varphi(\tau)]$, where*

$$\varphi(\tau)(z) = \frac{1}{(1-\tau^t z)^{n+1}}\,,$$

is an isometric imbedding of the unit ball $\mathbb{B}$ with the Bergman metric into $\mathbb{P}(A^2(\mathbb{B}^n))$ with the Fubini-Study metric. An automorphism g of $\mathbb{B}^n$ satisfies

$$\varphi(g(\tau))(z) = \varphi(\tau)(\hat{g}(z))J\hat{g}(z)\,, \tag{5.2}$$

with $\hat{g} = \bar{g}^{-1}$.

Proof. We establish first formula (5.2). As we will see, it is simply a consequence of the transformation formula of the Bergman kernel. Let g be given as in (5.1) and let $[X]$ denote the class of $X \in H$ in the projective space. Then

$$\varphi(g(\tau))(z) = \left[\frac{1}{(1-g(\tau)^t z)^{n+1}}\right] = \left[\frac{(C\tau+D)^{n+1}}{(\,(\tau^tC^t+D) - (\tau^tA^t+B^t)z\,)^{n+1}}\right]$$

$$= \left[\frac{1}{(\,\tau^t(C^t - A^tz) + (D-B^tz)\,)^{n+1}}\right] = \left[\frac{1}{\left(1-\tau^t\dfrac{A^tz-C}{-B^tz+D}\right)^{n+1}}\,\frac{1}{(-B^tz+D)^{n+1}}\right].$$

Formula (5.2) follows now from the fact that

$$\begin{pmatrix} A^t & -C^t \\ -B^t & D \end{pmatrix} = k \begin{pmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{pmatrix}^{-1} .$$

Since $(u_g f)(z) = f(\hat{g}(z))J\hat{g}(z)$ is a unitary operator in $A^2(\mathbb{B}^n)$, it projects to an isometry of $\mathbb{P}(A^2(\mathbb{B}^n))$, and therefore it is enough to prove that φ is an isometry at $\tau = 0$. To this effect, consider a differentiable curve $\tau(s)$ in $\mathbb{B}^n$ with $\tau(0) = (0, \ldots, 0)$ and $\tau'(0)$ a unit vector, say, $\tau'(0) = (1, 0, \ldots, 0)$. In the neighborhood of projective space with coordinates $(1, w_1, w_2, \ldots)$ we have

$$\varphi(\tau(s))(z) = \frac{1}{(1 - \tau^t(s)z)^{n+1}} = 1 + (n+1)\tau^t(s)z + \cdots .$$

At $s = 0$ this gives the tangent vector $(n+1)z_1$, for which the norm square is $(n+1)\pi^n/n!$, as required. This proves the theorem.

□

6. Imbedding of the generalized unit disk

The generalized unit disk Ω of degree n is the set of all $n \times n$ complex matrices W such that

$$I - W^t\overline{W} > 0 \quad , \quad W = W^t .$$

Siegel's metric is given by

$$ds^2 = \operatorname{tr}\left\{dW(I - W\overline{W})d\overline{W}(I - \overline{W}W)\right\} ,$$

where the group of isometries is the set of matrices

$$\begin{pmatrix} \overline{P} & \overline{Q} \\ Q & P \end{pmatrix}$$

satisfying $PQ^t = QP^t$, $\overline{P}P^t - \overline{Q}Q^t = I$.

The action we consider is

$$W^* = (-W\overline{Q} + P)^{-1}(W\overline{P} - Q) ,$$

with inverse

$$W = (W^*\overline{Q}^t + \overline{P}^t)^{-1}(W^*P^t + Q^t) .$$

As we shall see, this action will transfer to the usual action

$$Z^* = (\overline{P}Z + \overline{Q})(QZ + P)^{-1} ,$$

as [S].

The Bergman-Shilow boundary of Ω is the set of all symmetric unitary matrices of order n, that is,

$$S(\Omega) = \{Z \,:\, Z^*Z = I \,,\ Z^t = Z\} .$$

It has real dimension $n(n+1)/2$. Since it is not a group, it does not carry a natural Haar measure, but Bochner ([Bo], [K]) constructed a measure on $S(\Omega)$ as follows.

Let $\zeta_1, \dots, \zeta_k$ be an enumeration of the $k = n(n+1)/2$ entries of the symmetric unitary matrix (ζ_{pq}), and let

$$dV = \frac{d\zeta_1 \dots d\zeta_k}{\det(\zeta_{pq})^{(n+1)/2}} \cdot \frac{\Gamma(\frac{3}{2}) \dots \Gamma(\frac{n+1}{2})}{2^n \pi^{(n^2+3n)/2} i^{n(n+1)/2}} \,.$$

$L^2(S(\Omega), dV)$ is the space of all functions f square integrable with respect to the Bochner measure dV, and $H^2(S(\Omega), dV)$ is the subspace of functions admitting a holomorphic continuation to Ω. In the projective space $P(H^2)$ we shall consider the Fubini-Study metric.

Theorem 6.1. *The mapping $w \to [\varphi_w]$, where $\varphi_w(Z) = \det(I - WZ)^{-(n+1)/2}$, is an isometric imbedding of Siegel's generalized unit disk into the projective space*

$$\mathbb{P}(H^2(S(\Omega), dV)) \,.$$

The group of isometries of the disk is given by the restrictions to the imbedded surface of the (projection of) the unitary operators

$$\begin{aligned} (Uf)(z) &= f((\overline{P}Z + \overline{Q})(QZ + P)^{-1}) \\ &\qquad \det(QZ + P)^{-(n+1)/2} \,. \end{aligned}$$

(For even n the values of the determinants are obtained by analytic continuation from its initial value $+1$ for $Z = 0, P = I$.)

Proof. We first establish the action of the group of isometries on the image surface.

$$\varphi_{(-W\overline{Q}+P)^{-1}(W\overline{P}-Q)}(Z) = \det(I - (-W\overline{Q} + P)^{-1}(W\overline{P} - Q)Z)^{-(n+1)/2}$$

$$= \det(-W\overline{Q} + P)^{(n+1)/2} \det((-W\overline{Q} + P) - (W\overline{P} - Q)Z)^{-(n+1)/2}$$

$$= \det(-W\overline{Q} + P)^{(n+1)/2} \det((QZ + P) - W(\overline{P}Z + \overline{Q}))^{-(n+1)/2}$$

$$= \det(-W\overline{Q}+P)^{(n+1)/2} \det(I - W(\overline{P}Z+\overline{Q})(QZ+P)^{-1})^{(n+1)/2} \det(QZ+P)^{-(n+1)/2} \,.$$

Therefore, if we the denote by $[\varphi_W]$ the class of φ_W in $P(H^2)$, we obtain

$$[\varphi_{(-W\overline{Q}+P)^{-1}(W\overline{P}-Q)}(Z) = [\varphi_W](\overline{P}Z + \overline{Q})(QZ + P)^{-1}) \det(QZ + P)^{-(n+1)/2} \,.$$

Next, We prove that the operator $Uf(z)$ is unitary in H^2. We have that

$$\begin{aligned} \langle Uf, Ug \rangle &= \int_{S(\Omega)} f((\overline{P}Z + \overline{Q})(QZ + P)^{-1}) \overline{g((\overline{P}Z + \overline{Q})(QZ + P)^{-1}} \\ &\qquad \det(QZ + P)^{-(n+1)/2} \det \overline{(QZ + P)^{-(n+1)/2}} dV(Z) \,, \end{aligned}$$

and therefore, we have to show that if

$$\widehat{\zeta} = (\overline{P}\zeta + \overline{Q})(Q\zeta + P)^{-1}$$

then

$$\det(Q\zeta + P)^{-(n+1)/2}\det\overline{(Q\zeta + P)^{-(n+1)/2}}\frac{d\zeta \dots d\zeta_k}{\det(\zeta_{pq})^{(n+1)/2}} = \frac{d\widehat{\zeta} \dots d\widehat{\zeta}_k}{\det(\widehat{\zeta}_{pq})^{(n+1)/2}}\,. \tag{6.1}$$

On the one hand, we have

$$\begin{aligned} \widehat{\zeta}(Q\zeta + P) &= (\overline{P}\zeta + \overline{Q})\,, \\ d\widehat{\zeta}(Q\zeta + P) &= (\overline{P} - \widehat{\zeta}Q)d\zeta\,, \end{aligned}$$

and from $(Q\zeta + P)^t\widehat{\zeta} = (\overline{P}\zeta + \overline{Q})^t$ we get

$$(Q\zeta + P)^t d\widehat{\zeta}(Q\zeta + P) = d\zeta\,.$$

As they are forms of top degree it follows that

$$\det(Q\zeta + P)^{(n+1)}d\widehat{\zeta} \dots d\widehat{\zeta}_k = d\zeta \dots d\zeta_k$$

(see also the book by Hua, [Hu], and the observation by Bochner in [Bo] after formula (79) with $u = (Q\zeta + P)$). Since

$$\det(\widehat{\zeta}_{pq})^{(n+1)/2} = \det(\overline{P}\zeta + \overline{Q})^{(n+1)/2}\det(Q\zeta + P)^{-(n+1)/2}$$

we obtain

$$\frac{d\widehat{\zeta}_1 \dots d\widehat{\zeta}_k}{\det(\widehat{\zeta}_{pq})^{(n+1)/2}} = \frac{d\zeta_1 \dots d\zeta_k}{\det(\overline{P}\zeta + \overline{Q})^{(n+1)/2}\det(Q\zeta + P)^{(n+1)/2}}\,.$$

However

$$\begin{aligned} \det(\overline{Q\zeta + P}) &= \det(\overline{Q}\bar{\zeta} + \overline{P}) \\ &= \det(\overline{P}\zeta + \overline{Q})(\det\zeta)^{-1}\,, \end{aligned}$$

and the formula (5.1) is proved.

Since the action of the isometries on the unit disk gives isometries in $P(H^2)$, it is therefore enough to prove that the imbedding is an isometry at $W_0 = 0$. The metric in Siegel's disk at W_0 is

$$ds^2 \;=\; \mathrm{tr}(dW d\overline{W}) \;=\; \sum_{i,j} |dw_{ij}|^2\,.$$

If $W(t)$ is a curve with $W(0) = 0$ and $W'(0) = \dot{W}$, the length of the tangent vector is

$$\sum_{i,j} |w'_{ij}|^2\,.$$

On the image, the L^2 norm of the tangent vector is

$$\begin{aligned}\frac{d}{dt}\det(I-WZ)^{-(n+1)/2}|_{t=0} &= -\frac{n+1}{2}(\frac{d}{dt}\det(I-WZ))\det(I-WZ)^{-(n+3)/2}|_{t=0} \\ &= \frac{n+1}{2}\sum_{i,j} w'_{ij}Z_{ji}\,.\end{aligned}$$

The theorem will be proven as soon as we establish the following claim: the functions $(\sqrt{\frac{n+1}{2}}Z_{ij})$ are orthonormal in $H^2(\Omega, dV)$. To prove this we use the repoducing formula of Bochner

$$f(z) = \int_{S(\Omega)} \frac{f(\zeta)dV(\zeta)}{\det(I-Z\zeta^*)^{(n+1)/2}}\,.$$

Indeed, if we evaluate this formula at $f_{ij}(Z) = Z_{ij}$, we have

$$Z_{ij} = \int_{S(\Omega)} \frac{\zeta_{ij}dV(\zeta)}{\det(I-Z\zeta^*)^{(n+1)/2}}\,.$$

We now take the derivative $\partial/\partial Z_{kl}$ and evaluate at $Z=0$ to get

$$\delta_{ik}\delta_{jl} = \frac{n+1}{2}\int_{S(\Omega)} \zeta_{ij}\bar{\zeta}_{kL}dV(\zeta)$$

(observe that at $W_0 = 0$ the Fubini-study metric is just $h_{ij} = \delta_{ij}$).

$\square$

7. Imbedding of the hyperbolic three-space

We consider the upper-half-space

$$\mathbb{H}_3 = \{(x,y,t)\ :\ t>0\}$$

with the hyperbolic metric

$$ds^2 = \frac{dx^2+dy^2+dt^2}{t^2}\,.$$

The action of the hyperbolic group of isometries is described in terms of quaternions, so that if

$$\mathbb{H}_3 = \{q = x+iy+jt\,,:\ t>0\}$$

and $g \in PSL(2,\mathbb{C})$ then

$$g(q) = (aq+b)(cq+d)^{-1}\,.$$

Since this is a three dimensional real space, the methods devised by Calabi or Bieberbach do not apply directly for they were conceived for complex manifolds. However, the explicit imbeddings of the above sections lead to a formula which we now explain. On $L^2(\mathbb{C})$ there is a natural action of $PSL(2,\mathbb{C})$ via the formula

$$U_g f(z) = f(g(z)\,|g'(z)|\,.$$

This is a unitary operator, and the space we look for is the orbit of some funcion. The elements g in $PSL(2,\mathbb{C})$ that fix j correspond under the stereographic projection to rotations of the sphere; the area element of the sphere is invariant by them and therefore, on $L^2(\mathbb{C})$, we consider

$$F(z) = \frac{1}{1+|z|^2}\,.$$

This function satisfies $F(g(z))\,|g'(z)| = F(z)$ for those elements g that fix j. For $q \in \mathbb{H}_3$ we consider

$$g_q = \begin{pmatrix} \sqrt{t} & \dfrac{x+iy}{\sqrt{t}} \\ & \\ 0 & 1/\sqrt{t} \end{pmatrix}$$

so that $g_q(j) = q$. The imbedding we obtain is

$$q \to U_{g_q}\, F(z)\,.$$

Theorem 7.1. *The mapping $q \to F_q(z)$, where $q = x + iy + jt$,*

$$F_q(z) = \frac{\sqrt{6}t}{1+|zt+x+iy|^2},\; z \in \mathbb{C},$$

is an isometric imbedding of the hyperbolic three space into $L^2(\mathbb{C})$ with the usual Lebesgue measure. The group of isometries is given by the restriction of the unitary operators

$$U_g\, f(z) = f(g(z))\,|g'(z)|\,,\; g \in PSL(2,\mathbb{C})\,.$$

Proof. An arbitrary element in $PSL(2,\mathbb{C})$ may be decomposed into an element that fixes j (isomorphic to the rotation of the unit sphere) and an element g_q. U_{g_q} is an isometry of $L^2(\mathbb{C})$ carrying $F_j(z)$ to $F_q(z)$. To prove the theorem we have to check that the L^2 metric to the tangent space at F_j is equal to the hyperbolic length at j. Consider then a curve $\gamma(x) = (x(s), y(s), t(s))$ such that $\gamma(0) = j = (0,0,1)$ and $\gamma'(0) = (x', y', t')$. The image curve has tangent vector

$$F'(z) = \frac{t'}{1+|z|^2} - 2\frac{u_1(t'u_1 + x') + u_2(t'u_2 + y')}{(1+|z|^2)^2}\,,$$

where $z = u_1 + iu_2$. A direct computation gives the desired result:

$$\iint\limits_{\mathbb{R}^2} |R'(z)|^2\, du_1 du_2 = \frac{1}{6}\left((t')^2 + (x')^2 + (y')^2\right)\,.$$

This then explains the factor $\sqrt{6}$ which appears in the theorem and completes the proof. $\square$

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